ARTICLES

Electrostatic drift modes in a closed field line configuration

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The stability of electrostatic drift waves in a closed field line configuration in collisionality regimes ranging from collisional to collisionless is compared. The maximum sustainable pressure gradient is found to be dependent on the ratio of the temperature and density gradients ($\eta \equiv d \ln T/d \ln n$). The eigenmodes are seen to be flute-like. The stability boundary was found to be similar when both ions and electrons are collisional, when they are collisionless, and for collisional electrons and collisionless ions. The largest stable pressure gradients are obtained for $\eta \geqslant 2/3$. As the collisionality is reduced one observes some reduction of the region of stability. [DOI: 10.1063/1.1431594]

I. INTRODUCTION

Closed field line systems, such as a levitated dipole, provide a promising new approach for the magnetic confinement of plasmas for controlled fusion.[1,2] The plasma in a closed field line system can be stabilized in so-called ''bad curvature'' regions by plasma compressibility. In magnetohydrodynamic (MHD) theory stability of interchange modes limits the pressure gradient to a value $d \equiv d \ln p/d \ln U < \gamma$ with $U = \oint dl/B$ and γ the ratio of specific heats ($\gamma = 5/3$ in three-dimensional systems).

Recent studies which imposed a long mean free path collisional ordering (defined by $\omega_{bj} \gg \nu_j \gg \omega$, ω_{*j}, ω_{dj} with ω the wave frequency, ω_{bj} the bounce frequency, ν_j the collision frequency, ω_{*j} the diamagnetic drift frequency, ω_{dj} the curvature drift frequency) as compared with the short mean free path collisional MHD ordering ($\nu_j \gg \omega_{bj} \gg \omega$, ω_{*j}, ω_{dj}) to both electrons and ions[3–5] have shown that near marginal stability for MHD modes ($d \sim 5/3$) the MHD mode will couple to a potentially unstable drift frequency mode known as the ''entropy mode.'' The entropy mode is flute-like and stability depends on d and on the ratio $\eta = d \ln T/d \ln n$, with the most stable value being $\eta = 2/3$. Furthermore it has been shown that ion collisional relaxation (described as gyro-relaxation in Refs. 6,7) can destabilize the entropy mode and produce a weak instability in some otherwise stable regions of $d - \eta$ space.[5]

In a short mean free path collisional plasma (as characterized by MHD) the plasma compressibility derives from the local relationship of pressure to changing flux tube volume. The plasma is constrained to move with the field lines but in the case of closed field lines the field does not form flux surfaces and each closed field line must satisfy the equation of state. This imposes a periodicity on the perturbation that results in an important stabilizing term, the plasma compressibility. For open field lines the plasma is free to flow along field lines. This eliminates plasma compressibility.

For a collisionless plasma and considering low frequency perturbations that conserve the adiabatic invariants μ and J, particles that are magnetically trapped (all the particles for closed field lines) change energy as the field lines compress. This effect has been shown to correspond exactly to the MHD plasma compressibility. For passing particles, however, the passing particles sample the entire flux surface and the path integral of perturbed potential will average to zero.[8]

The description of drift ballooning modes in tokamaks, has been studied extensively.[9–11] The study of Ref. 11 indicates the importance of the parameter η in expanding the region of instability in a tokamak. However, the results in this reference apply to a tokamak equilibrium and in particular they do not apply to the case of zero safety factor and shear (i.e., $q = s = 0$).

In the collisionless regime, $\omega, \omega_{*j}, \omega_{dj} \gg \nu_j$, drift modes were shown to be stable when the MHD stability condition is met and $\eta \sim 2/3$[12] and stability was seen to improve as $k_\perp \rho_i$ increases, with $k_\perp$ the wave number perpendicular to the magnetic field direction and ρ_i the ion gyro radius. For $\eta > 2/3$ a drift frequency mode was observed and it became more unstable when $k_\perp \rho_i$ increased.

Electromagnetic effects on these modes have been studied. Wong *et al.* have shown that finite beta effects are stabilizing at low beta.[13] They also consider the electromagnetic effects at very high beta, and they have shown that there can be new regions of instability when $\beta \gg 1$.[13]

A levitated dipole experiment known as LDX is presently under construction.[2] The most relevant parameter regime for LDX would have collisional but long mean free path electrons ($\omega_{be} > \nu_e > \omega, \omega_{de}$) and collisionless ions. We will, therefore, consider the stability of low beta electrostatic modes with this intermediate collisional ordering and compare our results with those obtained with collisional and collisionless orderings.

II. SOLUTION OF DRIFT KINETIC EQUATION

We will compare the MHD prediction with the predictions of the more general plasma drift kinetic equation in the electrostatic limit. To compare the MHD result with kinetic theory we define

$$\hat{\omega}_{*p} \equiv \frac{\boldsymbol{b}\times\boldsymbol{k}\cdot\nabla p}{\Omega_i m_i n_i} \tag{1}$$

and

$$\hat{\omega}_d^{\mathrm{MHD}} \equiv \frac{2c}{e}\,\frac{Rk_\theta T}{1+\gamma\langle\beta\rangle/2}\,\frac{\oint dl\,\kappa/RB^2}{\oint dl/B}, \tag{2}$$

with R the cylindrical radial coordinate, κ the field line curvature, k_θ the azimuthal part of the perpendicular wave number ($k_\perp^2=k_\theta^2+k_R^2$), and $k_\theta R=m\gg 1$. One can show that d defined above is equal to $d=\hat{\omega}_{*p}/\hat{\omega}_d^{\mathrm{MHD}}$ and therefore the MHD stability requirement, $d\leqslant\gamma$, can be written as

$$\hat{\omega}_{*p}\leqslant\gamma\hat{\omega}_d^{MHD}. \tag{3}$$

We will consider the solution of the drift kinetic equation in the high collision frequency limit for electrons and low collision frequency for ions. We therefore apply the ordering for electrons:[14]

$$\Omega_{ce}>\omega_{be}>\nu_e>\omega_{*e}\sim\omega_{de}\sim\omega, \tag{4}$$

and for ions,

$$\Omega_{ci}>\omega_{bi}>\omega_{*i}\sim\omega_{di}\sim\omega>\nu_i, \tag{5}$$

with Ω_{cj} the appropriate cyclotron frequency.

To derive the stability criterion for electrostatic modes we consider a fluctuating potential (ϕ) and ignore any equilibrium electrostatic potential. From Faraday's law it is possible for a perturbation to leave the magnetic field undisturbed if $E=-\nabla\phi$ (which is consistent with $\beta\ll 1$).

The gyro kinetic equation was derived under the assumption that the wave frequency ω is less than the cyclotron frequency Ω_c and the perpendicular wavelength $\lambda=2\pi/k_\perp$ is short compared to a parallel wavelength, $2\pi/k_\parallel$. The appropriate equation for the gyro averaged distribution function $\tilde{f}$ is then[8,15,16]

$$\tilde{f}=q\phi F_{0\epsilon}+J_0(Z)h, \tag{6}$$

and the nonadiabatic response h satisfies

$$(\omega-\omega_d+iv_\parallel\boldsymbol{b}\cdot\nabla')h=-(\omega-\omega_*)q\phi F_{0\epsilon}J_0(Z)+iC(h). \tag{7}$$

In Eq. (7) $C(h)$ is the collision operator, $J_0(Z)$ is the Bessel function of the first kind, $F_0(\epsilon,\psi)$ is the equilibrium distribution function, i.e.,

$$F_0=\left(\frac{m}{2\pi T}\right)^{3/2} n_0 e^{-\epsilon/T}, \tag{8}$$

and

$$Z=\frac{k_\perp v_\perp}{\Omega_c},\quad F_{0\epsilon}\equiv\frac{\partial F_0}{\partial\epsilon},\quad \omega_*=\frac{\boldsymbol{b}\times\boldsymbol{k}\cdot\nabla' F_0}{m\Omega_c F_{0\epsilon}},$$

$$\omega_d=\boldsymbol{k}\cdot\boldsymbol{b}\times\frac{(v_\parallel^2\boldsymbol{b}\cdot\nabla\boldsymbol{b}+\mu\nabla B)}{\Omega_c}$$
$$=\boldsymbol{k}\cdot\boldsymbol{b}\times\frac{\epsilon(2(1-\lambda B)\boldsymbol{b}\cdot\nabla\boldsymbol{b}+\lambda\nabla B)}{\Omega_c}, \tag{9}$$
$$\boldsymbol{B}=\nabla\psi\times\nabla\theta,\quad \boldsymbol{b}=\boldsymbol{B}/|\boldsymbol{B}|.$$

We have defined $\epsilon=mv^2/2$, $\mu=mv_\perp^2/2B$, and $\lambda=\epsilon/\mu$. The prime on spatial gradients indicates that ϵ and μ are held fixed in the differentiation. The magnetic flux function is ψ and θ is the azimuthal angle. Assuming $k_\perp^2\rho_i^2\ll 1$,

$$J_0(Z)\approx 1-\frac{(k_\perp\rho_i)^2}{2}\,\frac{\lambda B_0^2}{B}\,\frac{\epsilon}{T_i}, \tag{10}$$

with the gyro radius ρ_i, defined as $\rho_i^2=T_i/m_i\Omega_{0i}^2$, B_0 the magnetic field at the field minimum at the location $R=R_0$ and Ω_0 the associated cyclotron frequency.

We consider perturbations whose growth time is long compared to a particle bounce time for all species and expand Eq. (7) in powers of $\delta=\omega/\omega_b$ with ω_b the bounce frequency, i.e., $h=h_0+\delta h_1+\delta^2 h_2+\cdots$. Notice $\delta\sim\omega_*/\omega_b\sim k_\theta\rho_i$ and corrections to the perturbed density will enter as $\delta^2\propto k_\theta^2\rho_i^2$. To zeroth order in δ we obtain from Eq. (7):

$$v_\parallel\boldsymbol{b}\cdot\nabla' h_0\approx 0, \tag{11}$$

i.e., $h_0=h_0(\epsilon,\mu,\psi)$ a constant along a field line. We determine the constant h_0 by taking the bounce average of the first order form of Eq. (7) to annihilate h_1 to obtain

$$(\omega-\overline{\omega_d})h_0=-(\omega-\omega_*)q\overline{\phi J_0}F_{0\epsilon}+i\overline{C}(h_0). \tag{12}$$

Since we will not pursue the ω/ω_b expansion to higher orders, we will suppress the subscript 0 in the following analysis of Eq. (12). The overbar indicates a bounce time average:

$$\overline{\phi}=\frac{\sqrt{m/2\epsilon}}{\tau_b}\oint\frac{\phi(l)\,dl}{\sqrt{1-\lambda B}}, \tag{13}$$

with the bounce time τ_b defined as $\tau_b=\sqrt{m/2\epsilon}\oint dl/\sqrt{1-\lambda B}$.

The electron collision operator conserves particles and energy. With the chosen ordering and these conservation properties the exact form of the electron collision operator does not enter the results.

Consider first collisional electrons following Ref. 3. We analyze Eq. (12) in the high collisionality limit ($\nu_e/\omega\gg 1$) and expand h_e in a subsidiary ordering, $h_e=h_0+h_1+\cdots$. To lowest order we find that h_0 is proportional to a Maxwellian distribution function, F_0, multiplied by a perturbed density, δN, and having a perturbed temperature $T+\delta T$.[14] Expanding to first order we obtain

$$h_{0e}=\delta N\left(\frac{m}{2\pi(T+\delta T)}\right)^{3/2} e^{-\epsilon/(T+\delta T)}$$
$$\approx\left[\frac{\delta N}{n_0}+\frac{\delta T}{T}\left(\frac{\epsilon}{T}-\frac{3}{2}\right)\right]F_0. \tag{14}$$

To next order the drift kinetic equation becomes

$$(\omega-\overline{\omega_d})h_0=-(\omega-\omega_*)q\overline{\phi J_0}F_{0\epsilon}+i\overline{C}(h_1). \tag{15}$$

We can obtain the perturbed density and temperature for the electron species by annihilating $\overline{C}(h_1)$ with the two operators:

$$\oint\frac{dl}{B}\int d^3v\begin{pmatrix}1\\ mv^2/2\end{pmatrix}$$
$$=\pi\left(\frac{2}{m}\right)^{3/2}\oint dl\int_0^\infty d\epsilon\begin{pmatrix}\epsilon^{1/2}\\ \epsilon^{3/2}\end{pmatrix}\int_0^{1/B}\frac{d\lambda}{\sqrt{1-\lambda B}}. \tag{16}$$

Since the collision operator conserves particles and energy these operators will annihilate it, i.e.,

$$\oint dl/B\int d^3v\,\overline{C}(h)=\oint dl/B\int d^3v\left(\frac{\epsilon}{T}-\frac{3}{2}\right)\overline{C}(h)=0. \tag{17}$$

We will define $\hat{\omega}_*$ by

$$\hat{\omega}_{*j}=\frac{T\boldsymbol{k}\times\boldsymbol{b}\cdot\boldsymbol{\nabla}n_0}{n_jm\Omega}, \tag{18}$$

and write

$$\omega_*=\hat{\omega}_*(1+\eta(\epsilon/T-3/2))F_0, \tag{19}$$

with $\eta=d\ln T/d\ln n$. Notice that $\hat{\omega}_{*p}=\hat{\omega}_*(1+\eta)$. Taking the flux tube and velocity space integral of Eq. (15) we obtain the following expression for the nonadiabatic perturbed density δN_e for electrons from Eq. (14):

$$\delta N_e=\frac{\hat{\omega}_{de}n_0(\delta T_e/T_e)+q_e(\omega-\hat{\omega}_{*e})\langle\phi\rangle/T_e}{\omega-\hat{\omega}_{de}}, \tag{20}$$

with the flux tube average defined as

$$\langle\phi\rangle=\frac{\int\phi dl/B}{\oint dl/B}, \tag{21}$$

and $\hat{\omega}_{dj}$, the flux tube averaged drift, defined as

$$\hat{\omega}_{dj}=\frac{cT_j(Rk_\theta)}{q_jU}\oint\frac{dl}{B^2R}(\kappa+\boldsymbol{\nabla}_\perp B/B), \tag{22}$$

with $U=\oint dl/B$. For low β, $\kappa\approx\boldsymbol{\nabla}_\perp B/B$ and $\hat{\omega}_d$ becomes equal to the MHD definition given in Eq. (2).

To obtain $(\delta T_e/T_e)$ we take the flux tube average and integrate over velocity space for $(\epsilon/T-3/2)\times$ Eq. (15) which again annihilates the collision operator to obtain

$$\frac{\delta T_e}{T_e}=\frac{\frac{2}{3}(\delta n_e/n_e)\hat{\omega}_{de}-q_e\eta_e\hat{\omega}_{*e}\langle\phi\rangle/T}{\omega-\frac{7}{3}\hat{\omega}_{de}}. \tag{23}$$

Equations (20) and (23) determine $\delta N_e/n$ and using Eq. (6) we construct the total electron density perturbation $\delta n_e/n=\delta N_e/n-q_e\phi/T_e$,

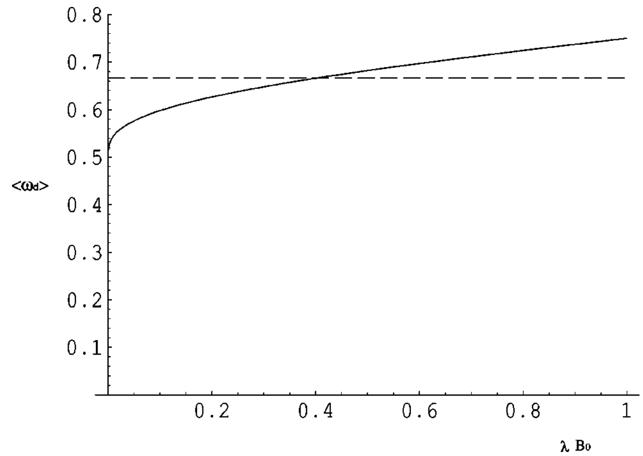

FIG. 1. Bounce averaged curvature drift frequency $\overline{\omega}_{di}$ normalized to $\epsilon\hat{\omega}_{di}/T$ [Eq. (22)] in a dipole field versus $B_0\lambda(=B_0/B_{\rm bounce})$.

$$\frac{\delta n_e}{n}=-\frac{q_e\phi}{T_e}+\frac{q_e\langle\phi\rangle}{T_e}\left[\frac{\omega^2-\omega(\frac{7}{3}\hat{\omega}_{de}+\hat{\omega}_{*e})+\hat{\omega}_{de}\hat{\omega}_{*e}(\frac{7}{3}-\eta_e)}{\omega^2-\frac{10}{3}\omega\,\hat{\omega}_{de}+\frac{5}{3}\hat{\omega}_{de}^2}\right]$$
$$\equiv\frac{q_e}{T_e}(-\phi+\langle\phi\rangle\Lambda_e^c(\omega,\hat{\omega}_{*e},\hat{\omega}_{de})). \tag{24}$$

If we assume both collisional electrons and ions so that the zero gyroradius ion response becomes similar to Eq. (24) (with $\hat{\omega}_{de},\hat{\omega}_{*e}\rightarrow\hat{\omega}_{di},\hat{\omega}_{*i}$) we can apply quasi-neutrality to obtain the marginal stability condition when

$$d=\frac{5}{7}\left[\frac{1+\eta}{1-\frac{3}{7}\eta}\right]. \tag{25}$$

A. Collisional electrons, collisionless ions

In Refs. 3–5 the ion response was assumed to be collisional. Here we assume a collisionless ion response and ignore terms of order $k_\perp^2\rho_i^2$ to obtain

$$\frac{\delta n_i}{n_i}=-\frac{q_i\phi}{T_i}+\frac{q_i}{T_i}\int d^3v\frac{\omega-\hat{\omega}_{*i}(1+\eta_i(\epsilon/T_i-3/2))}{\omega-\overline{\omega}_{di}(\epsilon,\lambda)}\overline{\phi}F_0$$
$$\equiv\frac{q_i}{T_i}(-\phi+\Lambda_i(\omega,\hat{\omega}_{*i},\hat{\omega}_{di})). \tag{26}$$

The pitch angle dependence of the normalized bounce average drift $\overline{\omega}_{di}(\epsilon,\lambda)T_i/(\epsilon\hat{\omega}_{di})$ is shown in Fig. 1 for the motion of a particle in the field of a point dipole. The pitch angle parameter λ varies in the range $(0<\lambda<1/B_0)$ with B_0 the minimum magnetic field on the outer midplane. We observe that $\overline{\omega}_{di}T_i/\epsilon\hat{\omega}_{di}$ is relatively constant as a function of pitch angle in a dipole field (it varies from 0.6 to 0.75) and to obtain the correct MHD response to order (ω_{di}/ω) when $\omega\gg\omega_*,\omega_d$ we will choose

$$\overline{\omega}_{di}(\epsilon,\lambda)\approx\frac{2}{3}\frac{\epsilon}{T_i}\hat{\omega}_{di}. \tag{27}$$

This approximation is discussed further in the Appendix. To obtain a dispersion relation we apply quasi-neutrality. Taking $T_i = T_e$ and applying quasi-neutrality to Eqs. (24) and (26) yields

$$2\phi = \Lambda_i + \langle \phi \rangle \Lambda_e^c. \tag{28}$$

Taking the flux tube average of Eq. (28) yields a dispersion relation that does not depend on the spatial dependence of the eigenfunction $\phi(l)$,

$$2 = F_i(\omega) + \Lambda_e^c, \tag{29}$$

where

$$F_i(\omega) = \int d^3 v F_0 \frac{\omega - \hat{\omega}_{*i}(1 + \eta_i(\epsilon/T_i - 3/2))}{\omega - \frac{2}{3}\frac{\epsilon}{T}\hat{\omega}_{di}}. \tag{30}$$

It has been shown for the collisional ordering that the solution is flute[5] and we will show below that, in the collisionless case the solution is also flute to order $k_\perp^2 \rho_i^2$. In the present intermediate collisionality case, a comparison of Eqs. (28) and (29) shows that a flute eigenmode is also present. Equation (29) can be written in the form

$$\begin{aligned}
&\Omega^3\left(\frac{3\,d\,\eta}{2} - 1 - \eta\right) I_1(\Omega) + \Omega^2\Bigg(1 + \eta - \frac{3\,d\,\eta}{2} \\
&\quad + \left(\frac{10}{3} - d + \frac{10\,\eta}{3} - \frac{7\,d\,\eta}{2}\right) I_1(\Omega)\Bigg) \\
&\quad + \Omega\Bigg(-\frac{13}{3}(1+\eta) + d + 5\,d\,\eta \\
&\quad + \left(-\frac{5}{3}(1+\eta) + \frac{10\,d}{3} - \frac{5\,d\,\eta}{2}\right) I_1(\Omega)\Bigg) \\
&\quad + \frac{10}{3}(1+\eta) - \frac{7\,d}{3} - \frac{3\,d\,\eta}{2} \\
&\quad + \left(\frac{-5\,d}{3} + \frac{5\,d\,\eta}{2}\right) I_1(\Omega) = 0,
\end{aligned} \tag{31}$$

with $\Omega = \omega/\hat{\omega}_{de}$, $d = (1+\eta)\omega_{*e}/\hat{\omega}_{de}$, and

$$I_1(y) = \frac{4}{\sqrt{\pi}} \int_0^\infty \frac{x^2 e^{-x^2} dx}{y + 2x^2/3}, \tag{32}$$

which can be expressed in terms of error functions. Equation (31) takes the form

$$D(\Omega) = \frac{d}{1+\eta}[F_1(\Omega) + \eta F_2(\Omega)] - F_3(\Omega) = 0. \tag{33}$$

It can be shown analytically that there is no marginally stable mode with $\Omega < 0$. For modes with $\Omega > 0$ there are no drift-resonant ions and, at marginal stability, there must be coincident real roots for Ω so that $\partial D/\partial \Omega = 0$, i.e.,

$$\frac{d}{1+\eta}[F_1'(\Omega) + \eta F_2'(\Omega)] = F_3'(\Omega). \tag{34}$$

Substituting Eq. (34) into Eq. (33) yields

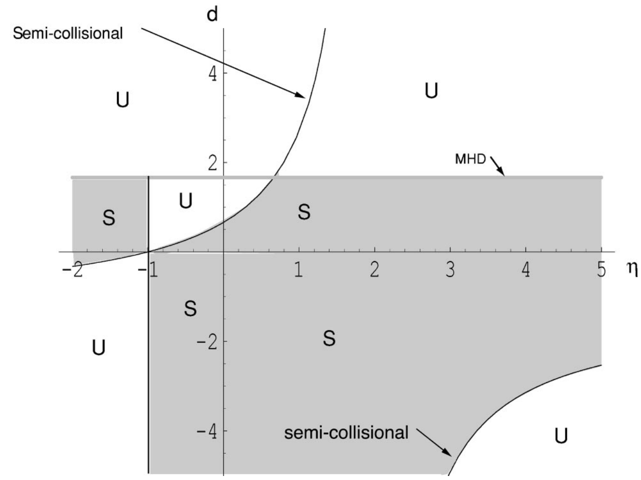

FIG. 2. The stability of semicollisional (collisional electron-collisionless ion) mode in (d, η). The stable region is shaded.

$$(F_1' F_3 - F_3' F_1) + \eta(F_2' F_3 - F_3' F_2) = 0. \tag{35}$$

One can show that $(F_2' F_3 - F_3' F_2) = -(3/2)(F_1' F_3 - F_3' F_1)$ and substituting into (35) yields the result

$$\left(1 - \frac{3}{2}\eta\right)(F_1' F_3 - F_3' F_1) = 0. \tag{36}$$

At marginal stability Eq. (36) yields a real frequency, independent of η,

$$\Omega_c = 0.3218. \tag{37}$$

Substituting $\Omega = \Omega_c$ into Eq. (33) yields the equation for the stability boundary:

$$d = 0.65868 \frac{1+\eta}{1 - 0.51214\eta}. \tag{38}$$

For $-1 < \eta < 2/3$ the stability boundary, Eq. (38), is close to, and below, the collisional boundary given by Eq. (25). At marginal stability the mode has a real frequency given by Eq. (37) and it propagates in the electron diamagnetic direction.

We have solved Eq. (29) numerically using a zero-finding routine to obtain the eigen frequency and we have utilized a Nyquist analysis to establish that we have found all possible unstable modes.

Figure 2 shows the stability diagram in $d - \eta$ space in the semi-collisional regime given by Eq. (38). The thick line at $d = 5/3$ is the MHD stability boundary. This appears when $k_\perp \rho_i \neq 0$ and the region $d > 5/3$ is unstable. Notice that at the point where the pressure gradient vanishes, $\eta = -1$ and $d = 0$, and this is a marginally stable point. The topology of the stability region is similar to that shown in Ref. 5 for the entropy mode.

We can add finite Larmor radius (FLR) terms in the limit $k_\perp \rho_i > k_\theta \rho_i$ (this limit does not require higher order terms in the bounce expansion). For sufficiently small FLR corrections a MHD-like mode appears when $d > 5/3$. For the collisional case it was found that increasing FLR terms will raise the stability boundary in the vicinity of $\eta = 2/3$ and reduce it elsewhere.[3] In the semi-collisional case discussed here we find that FLR corrections lead to a degradation of the stability boundary when $\eta > 2/3$. For example, for $\eta = 2$, $(k_\perp \rho_i)^2 = 0.001$ the stability boundary degrades by 5% (from $d = 5/3$). For $\eta < 2/3$ no significant degradation of the stability boundary is observed.

B. Collisionless mode

Rosenbluth[8] considered a collisionless isothermal plasma (i.e., $\eta = 0$) in a closed field line system. He showed that when $T_e = T_i$ an instability is always present, with zero real frequency at marginality, when d exceeds a critical value provided that some particles bounce in bad curvature. We now extend this result to arbitrary values of η, and obtain an analytic expression for the stability boundary in a point dipole [to be compared with Eqs. (25) and (38)].

To obtain a dispersion relationship for collisionless electrostatic modes we assume a collisionless electron response and apply quasi-neutrality. For $k_\perp^2 \rho_i^2 \ll 1$ we obtain the dispersion relation

$$2\phi = \int d^3v \, \frac{\omega - \omega_{*e}\left(1 + \eta_e\left(\frac{\epsilon}{T_e} - \frac{3}{2}\right)\right)}{\omega - \frac{2}{3}\frac{\epsilon}{T}\hat{\omega}_{de}} \overline{\phi} F_{0e}$$

$$+ \int d^3v \, \frac{\omega + \omega_{*e}\left(1 + \eta_i\left(\frac{\epsilon}{T_i} - \frac{3}{2}\right)\right)}{\omega + \frac{2}{3}\frac{\epsilon}{T}\hat{\omega}_{de}} \overline{\phi} F_{0i}. \quad (39)$$

Thus

$$2\phi = \frac{1}{2}(\Lambda_e^\epsilon + \Lambda_{i0}^\epsilon) \int \frac{B d\lambda}{\sqrt{1 - \lambda B}} \overline{\phi}, \quad (40)$$

where we have separated the energy and the pitch angle integrations. Taking the flux tube average (setting $T_e = T_i$, $n_e = n_i$) yields the dispersion relation

$$2 = \Lambda_e^\epsilon + \Lambda_{i0}^\epsilon, \quad (41)$$

which may be substituted into Eq. (40) to obtain

$$\phi - \frac{1}{2}\int \frac{B d\lambda}{\sqrt{1 - \lambda B}} \overline{\phi} = 0. \quad (42)$$

Multiplying Eq. (42) by ϕ and taking the flux tube average yields

$$\oint \frac{dl}{B} \int \frac{B d\lambda}{\sqrt{1 - \lambda B}} (\phi^2 - \overline{\phi}^2) = \int \tau_b d\lambda (\overline{\phi^2} - \overline{\phi}^2) = 0. \quad (43)$$

Since $\overline{\phi^2} - \overline{\phi}^2 \geq 0$ we can conclude that the mode is flute-like, i.e., $\phi = \phi_0$ to order $k_\perp^2 \rho_i^2$.

Introducing finite $k_\perp^2 \rho_i^2 \ll 1$ the general dispersion relation takes the form

$$-2\phi + \Lambda_i + \Lambda_e + k_\perp^2 \rho_i^2 \Lambda_{1i}(l) = 0, \quad (44)$$

where Λ_{1i} does depend on the arc length along the field line (l). Defining $t \equiv k_\perp^2 \rho_i^2 \ll 1$, we can expand $\phi = \phi_0 + t\phi_1 + \cdots$ and $\omega = \omega_0 + t\omega_1 + \cdots$. Then

$$-2\phi_0 + \Lambda_i(\phi_0) + \Lambda_e(\phi_0) = 0, \quad (45)$$

which we have shown requires the flute solution $\phi_0 \equiv \langle \phi_0 \rangle$. In first order,

$$-2\phi_1 + \Lambda_i(\phi_1) + \Lambda_e(\phi_1) + \Lambda_{1i}(l, \phi_0) + \frac{\partial \Lambda_i(\phi_0)}{\partial \omega}\omega_1$$

$$+ \frac{\partial \Lambda_e(\phi_0)}{\partial \omega}\omega_1 \overline{\phi^2} - \overline{\phi}^2 = 0. \quad (46)$$

Integrating $\oint dl/B$ annihilates the ϕ_1 terms and determines ω_1 from

$$\omega_1 \frac{\partial}{\partial \omega}[\Lambda_i(\omega_0) + \Lambda_e(\omega_0)] + \oint \frac{dl}{B}\Lambda_{1i}(l, \omega \equiv \omega_0) = 0. \quad (47)$$

This gives the shift in ω away from ω_0, caused by $k_\perp^2 \rho_i^2 \neq 0$. A similar perturbation analysis in the collisional and semi-collisional cases show that the low frequency electrostatic mode is flute-like, $\phi_0 = \langle \phi_0 \rangle$, in leading order of a $k_\perp^2 \rho_i^2 \ll 1$ expansion, in all collisionality regimes.

Returning to the leading order dispersion relation, Eq. (41) and following Ref. 8, we seek a solution with $\mathrm{Re}[\omega] = \mathrm{Im}[\omega] = 0$. Inserting this into Eq. (41) the velocity space integrals can be evaluated analytically to determine the stability boundary in the form

$$d = \frac{1}{3}\left[\frac{1 + \eta}{1 - \eta}\right]. \quad (48)$$

For $\eta = 0$, this recovers the stability result of Ref. 8; i.e., instability for $d > 1/3$. Figure 3 displays the stability diagram for the collisional, the collisional electron/collisionless ion and the collisionless modes. When $d > 5/3$ a vigorous MHD mode becomes unstable. For $\eta < 2/3$ an electron drift mode is responsible for a reduction of the stability limit (i.e., reduced d), whereas for $\eta > 2/3$ the stability boundary matches the MHD boundary, i.e., $d \approx 5/3$.

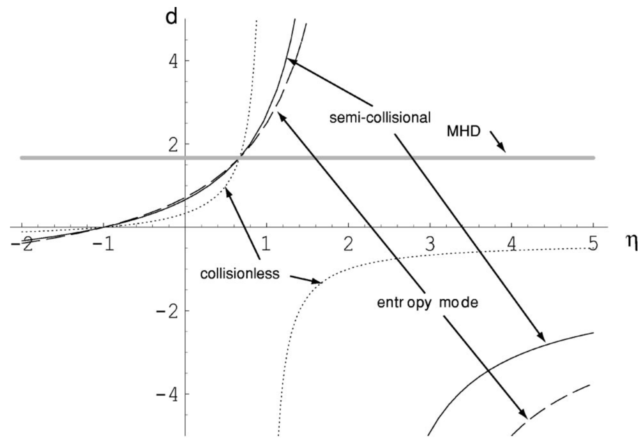

FIG. 3. The stability of the collisional entropy mode (dashed), the semicollisional mode (solid), and the collisionless mode (dotted) with $k_\perp^2 \rho_i^2 = 0$. The MHD boundary is also shown.

III. CONCLUSIONS

Two modes are seen to be present; a drift frequency mode with $\Omega \sim O(1)$ and a MHD mode with $\Omega \sim (k_\perp \rho_i)^{-1} \gg 1$. These modes are driven unstable by a combination of curvature and profile effects, characterized by the parameters d and η. We have analyzed the stability boundaries for these modes in the $d-\eta$ parameter space for varying values of plasma collisionality. We find a somewhat smaller stable region in this parameter space when collisionality is reduced. In the collisional regime the drift mode has been called the entropy mode.[5] In the collisionless regime the mode may be identified as an unfavorable curvature driven mode (for $d>0$)[8] or a temperature gradient driven mode (for $d<0$ and $\eta > \eta_{\mathrm{crit}}$).

In the LDX experiment we expect the parameters to be such that electrons are collisional but ions are collisionless ($n_e \sim 5\times 10^{12} - 10^{13}$ cm^{-3}, $T_e \sim T_i \sim 100-200$ eV). We find that as the ions become collisionless the stability boundary is somewhat degraded from the boundary determined by the entropy mode, without the destabilizing corrections that derive from collisional relaxation. In the collisionless case, which is relevant to the fusion reactor regime, we observe a further degradation of the stability boundary observed for $\eta < 2/3$.

In a dipole configuration, as described in Refs. 1,2, $\nabla T < 0, \nabla n_e < 0$, and therefore $\eta > 0$ in the region between the pressure peak and the wall. In this region we expect the pressure gradient to adjust so as to be just below the MHD limit, which for $\eta > 2/3$ indicates $d \lesssim 5/3$ and from Fig. 3 we expect to be in a drift stable regime. At the pressure peak $d=0$ and $\eta = -1$, which is a marginally stable point. On the contrary, in the region between the pressure peak and the internal coil $d<0$, and η can be positive or negative depending on the sign of ∇n_e (since $\nabla T > 0$) in this region. In this region close to the ring, weak, temperature gradient instabilities are possible, but Figs. 2 and 3 show that stable operation is possible for $-1 < \eta < \eta_{\mathrm{crit}}$, with η_{crit} of the order $1 \rightarrow 3$.

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APPENDIX: PITCH ANGLE VARIATION OF $\bar{\omega}_d$

Consider the collisionless case. From Fig. 1 we observe that a more accurate representation of the curvature drift term would be

$$\overline{\omega}_{di}(\epsilon,\lambda) \approx \frac{2}{3}\,\frac{\epsilon}{T_i}\hat{\omega}_{di}(1+\delta(\lambda B_{\mathrm{min}}-0.4)), \tag{A1}$$

and $\delta \sim 0.17$. Therefore, in the $\omega \equiv 0$ limit

$$\frac{\delta n_i}{n_i} = -\frac{q_i\phi}{T_i} + \frac{3}{2}\,\frac{d}{1+\eta}\int \frac{d^3v[1+\eta_i(\epsilon/T_i-3/2)]}{(\epsilon/T_i)[1+\delta(\lambda B_{\mathrm{min}}-0.4)]}\overline{q_i\phi}F_0/T_i\,. \tag{A2}$$

For $\delta \ll 1$ we obtain

$$\frac{\delta n_i}{n_i} \approx -\frac{q_i\phi}{T_i} + \frac{3d}{2}\,\frac{1-\eta}{1+\eta}\int \frac{B d\lambda}{\sqrt{1-\lambda B}}\,\overline{q_i\phi/T_i} \times (1-\delta(\lambda B_{\mathrm{min}}-0.4)), \tag{A3}$$

with a similar expression with $q_i \rightarrow q_e$ for electrons. Consider quasi-neutrality with $\phi = \phi_0 + \delta\phi_1(l) + \cdots$ and $d = d_0 + \delta d_1 + \cdots$ with ϕ_0 the flute solution. To lowest order the $\mathrm{Re}[\omega] = \mathrm{Im}[\omega] = 0$ mode defines a line $d_0 = d_0(\eta)$ [Eq. (48)] as before. The first order equation becomes

$$-\phi_1\frac{1+\eta}{1-\eta} + \frac{3d_0}{2}\int \frac{B d\lambda\,\overline{\phi_1(l)}}{\sqrt{1-\lambda B}} + \frac{3d_1\phi_0}{2}\int \frac{B d\lambda}{\sqrt{1-\lambda B}} - \frac{3d_0\phi_0}{2}\int \frac{B d\lambda(\lambda B_{\mathrm{min}}-0.4)}{\sqrt{1-\lambda B}} = 0. \tag{A4}$$

Multiplying by ϕ_0 and taking the flux tube average to annihilate $\bar{\phi}_1$ we obtain

$$\frac{d_1}{d_0} = \left(\frac{2}{3}\,\frac{B_{\mathrm{min}}\oint dl/B^2}{\oint dl/B} - 0.4\right). \tag{A5}$$

The flux tube integrals can be evaluated to give $d_1/d_0 \approx 0.06$. Thus

$$d = 1.01\frac{1}{3}\,\frac{1+\eta}{1-\eta}. \tag{A6}$$

We conclude that the variation of $\overline{\omega}_d$ with pitch angle drives ballooning structure, $\phi_1(l)$, in the eigen function but has a weak effect on the boundary in (d,η) space. For the weak variation in $\overline{\omega}_d(\lambda)$ in the point dipole, $\delta \approx 0.17$, the stability boundary is modified approximately 1%.

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